On decay-surge population models

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Deterministic population growth models

We take $\dot{x}_t = -\alpha(x_t)$, $x_0 = x \geq 0$.

- α is supposed to be continuous on $[0, \infty)$ and positive on $(0, \infty)$.
- With α_1, $a > 0$, consider the growth dynamics

$$\dot{x}_t(x) = -\alpha_1 x_t^a(x),\ x_0 = x, \tag{1}$$

for some growth field $\alpha(x) = -\alpha_1 x^a$. Integrating when $a \neq 1$, we get formally

$$x_t(x) = \left(x^{1-a} + \alpha_1 (a-1)\, t\right)^{1/(1-a)}. \tag{2}$$

1. The model
   - Piecewise deterministic Markov process (PDMP)
   - First jump distribution
   - Classification of state 0

2. Speed measure and Harris recurrence

Adding catastrophes

- Consider the stochastic process X_t that follows the deterministic flow with drift α
- β is the jump rate of the process and it depends on the position, β is a continuous function on $(0, \infty)$ and $\beta(x) > 0$, for all $x > 0$.

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Let

$$\mathbb{P}\left(X \geq y \mid X_{-} = x\right) = K\left(x, y\right), y \geq x,$$

be the kernel H which fixes the law of the jump amplitude.

In the separable case, $K\left(x, y\right) = \frac{k(y)}{k(x)}$ where k is non increasing function.

Piecewise deterministic Markov process (PDMP)

- Let $M(dt, dz)$ on $[0,\infty) \times [0,\infty)$ with intensity $dtdz$, a Poisson random measure.

- Let $(X_t)_{t\geq 0}$ be the PDMP obeying

$$dX_t = -\alpha(X_{t-})\,dt + \Delta(X_{t-}) \int_0^\infty \mathbf{1}_{\{z \leq \beta(X_{t-})\}} M(dt, dz),\ X_0 = x \geq 0 \quad (3)$$

This dynamics means alternatively that we have transitions

$$\begin{aligned} X_{t-} &= x \to x - \alpha(x)\,dt \text{ with probability } 1 - \beta(x)\,dt \\ X_{t-} &= x \to x + \Delta(x) \text{ with probability } \beta(x)\,dt. \end{aligned}$$

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- Notice that, between successive jumps the only possibility for the process to go up is by jumping.

Q: What is the law of the first jump?

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First jump distribution

Defining

$$T_x = \inf\{t > 0 : X_t \neq X_{t-} | X_0 = x\}, \inf \emptyset = \infty \tag{4}$$

the first jump time of the process. Introducing

$$\Gamma(x) := \int_1^x \gamma(y)\, dy, \text{ where } \gamma(x) := \beta(x)/\alpha(x), x > 0,$$

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- For $t < t_0(x) := \int_0^x \frac{dy}{\alpha(y)}$, we have

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds} = e^{-[\Gamma(x) - \Gamma(x_t(x))]}. \tag{5}$$

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Assumption 1

$\Gamma(0) = -\infty$.

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- If $t_0(x) < \infty$ and $\Gamma(0) > -\infty$, state 0 is accessible.
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- If $t_0(x) = \infty$ or $\Gamma(0) = -\infty$, state 0 is inaccessible.
- If $\beta(0) > 0$ and $K(0, y) = k(y)/k(0) > 0$ for some $y > 0$, state 0 is reflecting.
- If $\frac{\beta(0)}{k(0)} k(y) = 0$ for all $y > 0$, state 0 is absorbing.

1 The model

2 Speed measure and Harris recurrence
- Infinitesimal generator and speed measure
- Non explosion and recurrence

Infinitesimal generator

The associated infinitesimal generator is given for any smooth test function u by

$$Gu(x) = -\alpha(x)u'(x) + \beta(x)\int_x^\infty [u(y) - u(x)]K(x, dy), x \geq 0. \qquad (6)$$

In the separable case $K(x, y) = \frac{k(y)}{k(x)}$, the formula (6) is given by:

$$Gu(x) = -\alpha(x)u'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)u'(y)dy, \ x \geq 0.$$

Speed measure

- Suppose an invariant measure (or speed measure) $\pi\left(dy\right)$ exists.

- Since we supposed $\alpha(x) > 0$ for all $x > 0$, the explicit expression of the speed measure is given by $\pi(dy) = \pi(y)dy$ with

$$\pi\left(y\right) = C\frac{k\left(y\right)e^{\Gamma\left(y\right)}}{\alpha\left(y\right)} \tag{7}$$

Scale function

Definition 1

A scale function $s(x)$ of the process is any function solving $Gs(x) = 0$.

Assumption 2

Let

$$s(x) = \int_1^x \gamma(y) e^{-\Gamma(y)} / k(y) dy, \; x \geq 0, \tag{8}$$

and suppose that $s(\infty) = \infty$.

Non explosion

Let $S_1 < S_2 < \ldots < S_n < \ldots$ be the successive jump times of the process and $S_\infty = \lim_{n\to\infty} S_n$.

Proposition 1

Suppose $\Gamma(\infty) = \infty$ and suppose that Assumption 2 holds. Suppose also that β is continuous on $[0,\infty)$. Let V be any $\mathcal{C}^1-$function defined on $[0,\infty)$, such that $V(x) = 1 + s(x)$ on $[1,\infty)$ and such that $V(x) \geq 1/2$ for all x. Then V is a norm-like function in the sense of Meyn and Tweedie (1993), and we have

1. $GV(x) = 0, \, \forall x \geq 1.$
2. $\sup_{x\in[0,1]} |GV(x)| < \infty.$

As a consequence, $S_\infty = \sup_n S_n = \infty$ almost surely, so that X is non-explosive.

Recurrence of the process

- Suppose $\Gamma(\infty) = \infty$, such that non-trivial scale functions do exist.
- Assume that Assumption 2 holds since otherwise the process is either transient at ∞ or explodes in finite time.

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Theorem 2

Suppose we are in the separable case, that $k \in C^1$ and that 0 is inaccessible, that is, $t_0(x) = \infty$ for all x. Then every compact set $C \subset]0, \infty[$ is 'petite' in the sense of Meyn and Tweedie [4]. More precisely, there exist $t > 0$, $\alpha \in (0,1)$ and a probability measure ν on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, such that

$$P_t(x, dy) \geq \alpha \mathbf{1}_C(x) \nu(dy).$$

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Thank you for your attention!

Questions?