Déconvolution dans un modèle de régression en base d'Hermite

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Plan

1. Introduction
2. Estimation procedure
3. Rate of convergence
4. Illustration
5. Perspectives

Regression model

$$y(x_k) = h(x_k) + \varepsilon_k, \quad k = -n, \ldots, n-1, \tag{1.1}$$

where

$$h(x) = f \star g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy, \tag{1.2}$$

- fonction g : *kernel* is supposed known,
- $(x_k = kT/n)_{-n \leq k \leq n-1}$ where $0 < T < \infty$, fixed,
- $(\varepsilon_k)_{-n \leq k \leq n-1}$ (noise) i.i.d. with $\mathbb{E}[\varepsilon_k] = 0$ and $\mathrm{Var}(\varepsilon_k) = \sigma_\varepsilon^2 < \infty$, known,
- f is the unknown function to be estimated.

Special cases !

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Special cases !

Some references

If $\text{supp(f)} \cap \text{supp(g)} \subset (0, +\infty($:

- Del *et al* (1998) study the problem for $g(x) = be^{-ax}1_{x\geq 0}$ and using linear differential equation.
- Abramovich *et al* (2013) summarize the problem to estimating $h^{(d)}$ by a kernel method.
- Comte *et al* (2017) propose a projection method in the Laguerre basis.

Goal : extend theses results to the case $f \neq 0$ and $g \neq 0$ on $)-\infty, 0)$.

Two estimations methods are considered :

- **deconvolution-projection procedure,**
- projection-projection procedure.

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Hermite basis

Define the Hermite basis $(\varphi_j)_{j\geq 0}$ from Hermite polynomials $(H_j)_{j\geq 0}$:

$$\varphi_j(x) = c_j H_j(x) e^{-x^2/2}, \quad H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j}(e^{-x^2}), \quad c_j = (2^j j! \sqrt{\pi})^{-1/2}, \quad j \geq 0.$$

The Hermite polynomials $(H_j)_{j\geq 0}$:

$$\int_{\mathbb{R}} H_j(x) H_k(x) e^{-x^2} dx = 2^j j! \sqrt{\pi} \delta_{j,k}.$$

Note that :

$$\varphi_j^* = \sqrt{2\pi}(i)^j \varphi_j$$

and

$$|\varphi_j(x)| \leq C e^{-\zeta x^2}, \quad |x| \geq \sqrt{2j+1} \quad C, \zeta > 0.$$

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Assumptions

(**A**1) f and f^* belong to $\mathbb{L}^1(\mathbb{R})$ where $t^*(u) = \int e^{iux} t(x)dx$ denotes the Fourier transform of t.

(**A**2) $g^* \neq 0$.

(**A**3) There exists $c_1 \geq c_1' > 0$, $\gamma > 0$, such that

$$c_1'(1+t^2)^{\gamma} \leq |g^*(t)|^{-2} \leq c_1(1+t^2)^{\gamma}, \quad \forall t \in \mathbb{R}.$$

We have :

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{h^*(u)}{g^*(u)} du, \quad \forall x \in \mathbb{R}.$$

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Estimator of f

Set :

$$\Phi_d = (\varphi_j(x_k))_{-n \le k \le n-1, 0 \le j \le d-1}, \quad \Psi_d = \frac{T}{n} \Phi_d^t \Phi_d.$$

The projection estimator of h :

$$\widehat{h}_d = \sum_{j=0}^{d-1} \widehat{a}_j^{(d)} \varphi_j,$$

$$\vec{\widehat{a}}^{(d)} := \begin{pmatrix} \widehat{a}_0^{(d)} \\ \vdots \\ \widehat{a}_{d-1}^{(d)} \end{pmatrix} = \left(\Phi_d^t \Phi_d\right)^{-1} \Phi_d^t \vec{\mathscr{Y}} = \frac{T}{n} \Psi_d^{-1} \Phi_d^t \vec{\mathscr{Y}}, \qquad \vec{\mathscr{Y}} = \begin{pmatrix} (y(x_{-n}) \\ \vdots \\ y(x_{n-1}) \end{pmatrix}.$$

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$$\widehat{f}_{(\ell),d}(x) = \frac{1}{2\pi} \int_{-\ell}^{\ell} e^{-iux} \frac{\widehat{h}_d^*(u)}{g^*(u)} du, \quad \ell > 0.$$

Performance of $\widehat{f}_{(\ell),d}$

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Performance of $\widehat{f}_{(\ell),d}$

Risk bound of $\widehat{f}_{(\ell),d}$

Set

$$\Delta(\ell) = \sup_{|u|\leq \ell} |g^*(u)|^{-2}, \qquad f_{(\ell)}(x) = \frac{1}{2\pi}\int_{-\ell}^{\ell} e^{-iux} f^*(u)du$$

Proposition

*Under (**A**1) and (**A**2), we have*

$$\mathbb{E}\left[\|\widehat{f}_{(\ell),d} - f\|^2\right] \leq \|f - f_{(\ell)}\|^2 + \Delta(\ell)\left(\|h - h_d\|^2 + \lambda_{max}\left(\Psi_d^{-1}\right)\|h - h_d\|_n^2 + \sigma_\varepsilon^2 \frac{T}{n}\mathrm{tr}\left(\Psi_d^{-1}\right)\right).$$

- The first term ($\|f - f_{(\ell)}\|^2 = \frac{1}{2\pi}\int_{|u|>\ell}|f^*(u)|^2 du$) is the classical bias term.
- The term $\Delta(\ell)$ corresponds to the deconvolution aspect of problem.
- The terms in the big parenthesis represent the regression aspect of problem.

Rate of convergence ?

Risk bound of $\widehat{f}_{(\ell),d}$

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Rate of convergence ?

Regularity spaces

Definition (Sobolev-Hermite ball)

Let $s > 0$ and $L > 0$, define the Sobolev-Hermite ball of regularity s by :

$$W_H^s(L) = \{\theta \in \mathbb{L}^2(\mathbb{R}), \quad \sum_{k\geq 0} k^s a_k^2(\theta) \leq L\}, \quad a_k(\theta) = \int_{\mathbb{R}} \theta(x)\varphi_k(x)dx.$$

Definition (Sobolev ball)

Recall also that the usual Sobolev ball $W^s(L)$ is defined, for $s > 0$ by

$$W^s(L) = \{\theta \in \mathbb{L}^2(\mathbb{R}), \int (1+u^2)^s |\theta^*(u)|^2 du < L\}.$$

Note that (see Bongioanni and Torrea (2006))

- $W_H^s(L) \subset W^s(L^*)$
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Additional Assumption :

(**A**4) There exists $\lambda > 0$ such that : $\lambda_{max}(\Psi_d^{-1}) \leq \lambda < \infty$ uniformly in d.

Theorem (Rate on Sobolev ball)

*Let assumptions (**A**1) to (**A**4) hold and $h \in W_H^{s+\gamma}(L)$. For $d_{opt} = [n^{1/(s+\gamma+1)}]$ with $s+\gamma > 11/6$ and $\ell_{opt} \propto n^{1/2(s+\gamma+1)}$, we derive that*

$$\sup_{f \in W^s(L)} \mathbb{E}[\|\widehat{f}_{(\ell_{opt}),d_{opt}} - f\|^2] = \mathcal{O}\left(n^{-\frac{s}{s+\gamma+1}}\right).$$

- $\widehat{f}_{(\ell_{opt}),d_{opt}}$ converges at a polynomial rate as in density deconvolution for ordinary smooth noise (see Fan (1991)).
- This rate is not standard and it is specific to the Hermite basis (regression part).

Choices d_{opt} and ℓ_{opt} are not feasible in practice ?

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Adaptive for $\widehat{f}_{(\ell),d}$ with the GL method

Set $\ell = \sqrt{2d}$.

$$\widetilde{f}_{(d)}(x) := \widehat{f}_{(\sqrt{2d}),d}(x) = \frac{1}{2\pi}\int_{-\sqrt{2d}}^{\sqrt{2d}} e^{-iux}\frac{\widehat{h}_d^*(u)}{g^*(u)}du.$$

We estimate the bias by :

$$\widetilde{A}(d) := \max_{d' \in \mathscr{M}_n}\left\{\left(\|\widetilde{f}_{d'} - \widetilde{f}_{d\wedge d'}\|^2 - \kappa_1 V(d')\right)_+\right\}, \quad \kappa_1 > 0$$

$$V(d) = 2(1+24\log(n))\sigma_\varepsilon^2 \Delta(\sqrt{2d})\frac{\lambda dT}{n}.$$

We select $\widetilde{d}$:

$$\widetilde{d} := \arg\min_{d \in \mathscr{M}_n}\left\{\widetilde{A}(d) + \kappa_2 V(d)\right\},$$

where $\kappa_1 \leq \kappa_2$ and $\mathscr{M}_n$ a finite model collections.

κ_1 and κ_2 must be calibrated !

Oracle inequalities

Theorem

*Under (**A**0) to (**A**3) and ε sub-Gaussian, for $\kappa_1 \geq 12$,*

$$\mathbb{E}[\|\widetilde{f}_{(\widetilde{d})} - f\|^2] \leq C \inf_{d \in \mathcal{M}_n} \left(\|f - f_{(\sqrt{2d})}\|^2 + R_b(d) + V(d) \right) + C' \frac{\log(n)}{n},$$

where

$$R_b(d) := \max_{d' \in \mathcal{M}_n, d \leq d'} \left(\Delta(\sqrt{2d'}) \|h - \mathbb{E}[\widehat{h}_{d'}]\|^2 \right).$$

If $f \in W_H^s(L)$ and $h \in W_H^{s+\gamma}(L')$ with $s + \gamma \geq 17/6$,

$$\mathbb{E}[\|\widetilde{f}_{(\widetilde{d})} - f\|^2] \leq C_1 \inf_{d \in \mathcal{M}_n} \left(d^{-s} + V(d) \right) + C_1' \frac{\log(n)}{n},$$

- The two inequalities show that $\widetilde{f}_{(\widetilde{d})}$ realizes automatically a bias-variance trade-off.
- The lower bound on κ_1 is enough large.
- The variance σ_ε^2 must be estimated.

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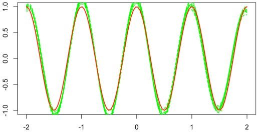
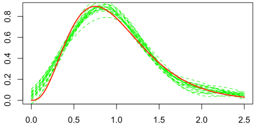
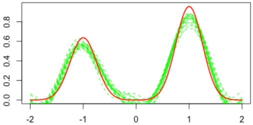
Figures



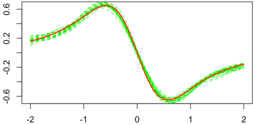
$\overline{\widetilde{d}} = 27$, $\overline{s2n} = 2.64$



$\overline{\widetilde{d}} = 13.85$, $\overline{s2n} = 1.60$



$\overline{\widetilde{d}} = 20.05$, $\overline{s2n} = 1.18$



$\overline{\widetilde{d}} = 12.65$, $\overline{s2n} = 1.87$

FIGURE – 20 estimates of $\widetilde{f}_{(\widetilde{d})}$. The true function is in bold red and the estimates in green dotted lines for $n = 1000$.

Perspectives

1. **Conclusion**
   - Deconvolution estimator is proposed and upper bounds are proved
   - Adaptive procedure and oracle inequalities are proved
   - Non asymptotic result
2. **Perspectives**
   - Optimality
   - Extend the result for random designs
   - Consider the multivariate functions

Thank you for your attention !

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2. **Perspectives**
   - Optimality
   - Extend the result for random designs
   - Consider the multivariate functions

Thank you for your attention !

Perspectives

1. **Conclusion**
   - Deconvolution estimator is proposed and upper bounds are proved
   - Adaptive procedure and oracle inequalities are proved
   - Non asymptotic result
2. **Perspectives**
   - Optimality
   - Extend the result for random designs
   - Consider the multivariate functions

Thank you for your attention !

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